

Banco de México
Documentos de Investigación

Banco de México
Working Papers

N° 2006-13

**Estimating Integrated Volatility Using Absolute
High-Frequency Returns**

Carla Ysusi
Banco de México

December 2006

La serie de Documentos de Investigación del Banco de México divulga resultados preliminares de trabajos de investigación económica realizados en el Banco de México con la finalidad de propiciar el intercambio y debate de ideas. El contenido de los Documentos de Investigación, así como las conclusiones que de ellos se derivan, son responsabilidad exclusiva de los autores y no reflejan necesariamente las del Banco de México.

The Working Papers series of Banco de México disseminates preliminary results of economic research conducted at Banco de México in order to promote the exchange and debate of ideas. The views and conclusions presented in the Working Papers are exclusively the responsibility of the authors and do not necessarily reflect those of Banco de México.

Estimating Integrated Volatility Using Absolute High-Frequency Returns*

Carla Ysusi[†]

Banco de México

Abstract

When high-frequency data is available, in the context of a stochastic volatility model, realised absolute variation can estimate integrated spot volatility. A central limit theory enables us to do filtering and smoothing using model-based and model-free approaches in order to improve the precision of these estimators. Although the absolute values are empirically attractive as they are less sensitive to possible large movements in high-frequency data, realised absolute variation does not estimate integrated variance. Some problems arise when using a finite number of intra-day observations, as explained here.

Keywords: Quadratic variation, Absolute variation, Stochastic volatility models, Semimartingale, High-frequency data.

JEL Classification: C13, C51, G19

Resumen

Bajo modelos de volatilidad estocástica, la volatilidad spot integrada puede ser estimada con la variación absoluta realizada utilizando datos en alta frecuencia. Dadas las distribuciones asintóticas, la precisión de estos estimadores puede mejorarse a través de filtrado y suavizamiento. A pesar de que el uso de valores absolutos es empíricamente atractivo dado que son menos sensibles a posibles valores extremos, la variación absoluta realizada no es un estimador de la varianza integrada. Diferentes problemas pueden presentarse al usar un número finito de observaciones intra-día como se explica en este documento.

Palabras Clave: Variación cuadrática, Variación absoluta, Modelos de volatilidad estocástica, Semimartingala, Datos en alta frecuencia.

*I am grateful to Neil Shephard for his constant advice and support. I wish to thank Valentina Corradi and Matthias Winkel for their helpful suggestions.

[†] Dirección General de Investigación Económica. Email: cysusi@banxico.org.mx.

1. Introduction

Prices have been recently thought as realisations of continuous time diffusion processes. Complete records of prices are available for many financial assets at a high-frequency, therefore continuous time models can be calibrated. More precisely, within a continuous semimartingale process the sum of high-frequency squared returns estimates quadratic variation. This is why realised variance (the sum of finite squared intra-day returns) can be used as an estimator of integrated variance in a stochastic volatility model. This result has been extensively studied in the recent literature, e.g. Andersen and Bollerslev (1998a), Barndorff-Nielsen and Shephard (2002) and Comte and Renault (1998). However this is an asymptotic result and infinitesimal returns do not occur in real life. An asymptotic theory of estimation error was developed to distinguish between true underlying variability and measurement noise.

Realised variance has been used in financial econometrics for many years, examples include Rosenberg (1972), Merton (1980), Poterba and Summers (1986), Schwert (1989), Hsieh (1991), Zhou (1996), Taylor and Xu (1997), Christensen and Prabhala (1998), Andersen, Bollerslev, Diebold and Ebens (2001), Andersen, Bollerslev, Diebold and Labys (2001). Recent literature using quadratic variation for semimartingales has been the independently and concurrently development by Andersen and Bollerslev (1998a), Comte and Renault (1998) and Barndorff-Nielsen and Shephard (2001). Many other papers on realised variance exist which are discussed in Dacorogna et al. (2001) and in the reviews by Andersen, Bollerslev and Diebold (2005) and Barndorff-Nielsen and Shephard (2007).

An alternative approach to the realised variance would be using the sum of the absolute value of the increments of the intra-day log-prices, named realised absolute variation, in order to estimate the integrated spot volatility of a stochastic volatility model. This is empirically attractive for using absolute values is less sensitive to possible large movements in high-frequency data. Taylor (1986) and Ding, Granger and Engle (1993) recognized that empirically absolute returns are more persistent than squared returns. Andersen and Bollerslev (1997) and Andersen and Bollerslev (1998b) empirically studied the properties of realised absolute variation of speculative assets, nevertheless the approach was abandoned in their subsequent work due to the lack of appropriate theory. Ghysels, Santa-Clara and Valkanov (2003) and Forsberg and Ghysels (2006) retake the interest in absolute returns and provide empirical and theoretical explanations for the outperformance of realised absolute variation.

Although its empirical advantages, absolute returns (realised absolute variation) have not been thoroughly studied. Therefore, in this paper, given a stochastic volatility model where the log-prices are a continuous semimartingale, realised absolute variation is used to estimate the integrated spot volatility. Following Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004), the corresponding limit theory will enable us to do filtering and smoothing using a model based and model free approach to improve the precision of the estimators.

When an underlying continuous process is assumed for the log-prices, the use of high frequency data to measure volatility can give misleading results because discrete observations are contaminated

by market microstructure effects. Although the efficiency of realised absolute variation is measured with asymptotic results, using data at the highest available frequency will not necessarily be the best approach. Nevertheless, in this paper, we will assume that our observations are not affected by market microstructure noise in order to reach some conclusions; further work is needed to asses the effect of this noise in the estimations.

The outline of this paper is as follows. In Section 2 we present some definitions and results for realised variance and realised absolute variation given in the main literature. We apply the methods of estimation proposed in Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004), in Section 3, to realised absolute variation. Finally in Section 4 an analysis about the benefits and faults of using realised absolute variation is done following the results in Forsberg and Ghysels (2006). We conclude in section 5 and a description of the dataset is given in the Appendix.

2. Framework and properties

A standard model in financial economics is a stochastic volatility (SV) model for log-prices Y_t which follows the equation

$$Y_t = \int_0^t a_u du + \int_0^t \sigma_s dW_s, \quad t \geq 0, \quad (1)$$

where we denote $A_t = \int_0^t a_u du$. The processes σ_t and A_t are assumed to be stochastically independent of the standard Brownian motion W_t . Here σ_t is called the instantaneous or spot volatility, σ_t^2 the corresponding spot variance and A_t the mean process. A simple example of this process is

$$A_t = \mu t + \beta \sigma_t^{2*}, \quad \text{where} \quad \sigma_t^{2*} = \int_0^t \sigma_s^2 ds.$$

The process σ_t^{2*} is called the integrated variance.

More generally A_t is assumed to have continuous locally bounded variation paths and it is set that $M_t = \int_0^t \sigma_s dW_s$, with the added condition that $\int_0^t \sigma_s^2 ds < \infty$ for all t . This is enough to guarantee that M_t is a local martingale. So the original equation (1) can be rewritten as

$$Y_t = A_t + M_t.$$

Under these assumptions Y_t is a continuous semimartingale (see Protter (1990)).

It is essential to define a discretised version of Y_t based on intervals of time of length $\delta > 0$. Given the previous framework, let the δ -returns be

$$y_j = Y_{j\delta} - Y_{(j-1)\delta} \quad j = 1, 2, 3, \dots, \lfloor t/\delta \rfloor.$$

Then, independently of the model for the volatility, if A_t and σ_t are stochastically independent of W_t , this implies that

$$y_j | a_j, \nu_j^2 \sim N(a_j, \nu_j^2) \quad (2)$$

where $a_j = A_{j\delta} - A_{(j-1)\delta}$ and $\nu_j^2 = \sigma_{j\delta}^{2*} - \sigma_{(j-1)\delta}^{2*}$. Usually a_j is called the actual mean and ν_j^2 the actual variance.

One of the most important aspects of semimartingales is the quadratic variation (QV), defined as

$$[Y]_t = p \lim_{n \rightarrow \infty} \sum_{j=1}^n (Y_{t_j} - Y_{t_{j-1}})^2,$$

for any sequence of partitions $t_0^{(n)} = 0 < t_1^{(n)} < \dots < t_n^{(n)} = t$ with $\sup_j \{t_j^{(n)} - t_{j-1}^{(n)}\} \rightarrow 0$ as $n \rightarrow \infty$. As A_t is assumed to be continuous and of finite variation we obtain that

$$[Y]_t = [A]_t + 2[A, M]_t + [M]_t = [M]_t = \int_0^t \sigma_u^2 du$$

where

$$[X, Y]_t = p \lim_{n \rightarrow \infty} \sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})(Y_{t_j} - Y_{t_{j-1}}).$$

This holds since the quadratic variation of any continuous, locally bounded variation process is zero (see Protter (1990)).

The generalisation of quadratic variation is

$$[Y]_t^{[r]} = p \lim_{n \rightarrow \infty} \sum_{j=1}^n |Y_{t_j} - Y_{t_{j-1}}|^r,$$

for any sequence of partitions $t_0^{(n)} = 0 < t_1^{(n)} < \dots < t_n^{(n)} = t$ with $\sup_j \{t_j^{(n)} - t_{j-1}^{(n)}\} \rightarrow 0$ as $n \rightarrow \infty$.

Now, based on the spot volatility σ_t , we can define alternatively the integrated spot volatility

$$\sigma_t^{[1]*} = \int_0^t \sigma_s ds$$

and the actual volatility

$$\nu_j^{[1]} = \sigma_{j\delta}^{[1]*} - \sigma_{(j-1)\delta}^{[1]*} \quad j = 1, 2, 3, \dots, \lfloor t/\delta \rfloor.$$

It is important to notice that the actual volatility is different from the square root of the actual variance, so $(\nu_j^{[1]})^2 \neq \nu_j^2$.

More about SV models can be found in, for example, Shiryaev (1999) or Shephard (2005).

2.1. Realised absolute variation

The realised absolute variation process is defined as

$$[Y_\delta]_t^{[1]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} |y|_j.$$

Barndorff-Nielsen and Shephard (2003) derived a limit theorem for the realised power variation

process

$$[Y_\delta]_t^{[r]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} |y_j|^r,$$

as $\delta \downarrow 0$. Given that realised absolute variation is a special case of realised power variation, $r = 1$, the asymptotic result can be used to study the properties of the estimation error of integrated spot volatility.

Barndorff-Nielsen and Shephard (2003) gave the Central Limit Theorem for the realised power variation process with some restrictive conditions. Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) provide some general limit results for realised power and bipower variation which are proved under much weaker assumptions.

For the SV model (1) where A_t is of locally bounded variation, $\int_0^t \sigma_u^2 du < \infty$ and σ_t is càdlàg, we have that for $\delta \downarrow 0$,

$$\mu_r^{-1} \delta^{1-r/2} [Y_\delta]_t^{[r]} \xrightarrow{p} \int_0^t \sigma_s^r ds$$

and

$$\frac{\mu_r^{-1} \delta^{1-r/2} [Y_\delta]_t^{[r]} - \int_0^t \sigma_s^r ds}{\mu_r^{-1} \delta^{1-r/2} \sqrt{\mu_{2r}^{-1} v_r [Y_\delta]_t^{[2r]}}} \xrightarrow{L} N(0, 1),$$

where $\mu_r = E(|X|^r)$ and $v_r = \text{Var}(|X|^r)$ with $X \sim N(0, 1)$.

In particular, this implies that

$$\frac{d[Y]_t^{[1]}}{dt} = \sigma_t.$$

From these, we can obtain the relevant result for our case ($r = 1$) where

$$\frac{\sqrt{\delta}}{\mu_1} [Y_\delta]_t^{[1]} \xrightarrow{p} [Y]_t^{[1]}$$

where $\mu_1 = \sqrt{2/\pi}$, and where

$$\frac{\frac{\sqrt{\delta}}{\mu_1} [Y_\delta]_t^{[1]} - [Y]_t^{[1]}}{\sqrt{\delta(\mu_1^{-2} - 1)[Y_\delta]_t^{[2]}}} \xrightarrow{L} N(0, 1).$$

For daily series, suppose there are $\lfloor t/\delta \rfloor = M$ intra- h observations during each fixed h time period (here h denotes the period of a day) defined as

$$y_{j,i} = Y_{(i-1)h+j\delta} - Y_{(i-1)h+(j-1)\delta},$$

for the $j - th$ intra- h return for the $i - th$ period. Then the realised absolute variation, i.e. a scaled sum of the absolute value of the intra-day changes of log-prices, will be defined here as

$$[Y_M]_i^{[1]} = \frac{1}{\sqrt{M}\mu_1} \sum_{j=1}^M |y_{j,i}|$$

and therefore, $[Y_M]_i^{[1]} \rightarrow \sigma_{ih}^{*[1]} - \sigma_{(i-1)h}^{*[1]}$ when $M \rightarrow \infty$. So, $\widehat{\nu}_i^{[1]} = [Y_M]_i^{[1]}$ converges to $\nu_i^{[1]}$.

Consequently we can use the realised absolute variation as an estimator of the actual volatility.

Using the intra-day INTEL prices, the realised absolute variation using five and sixty minutes returns are plotted, as well as their correspondent autocorrelation functions (Figure 1). As M gets smaller, the series become more jagged. Also, the autocorrelation is higher when the value of M is big, and it shows a slower decay.

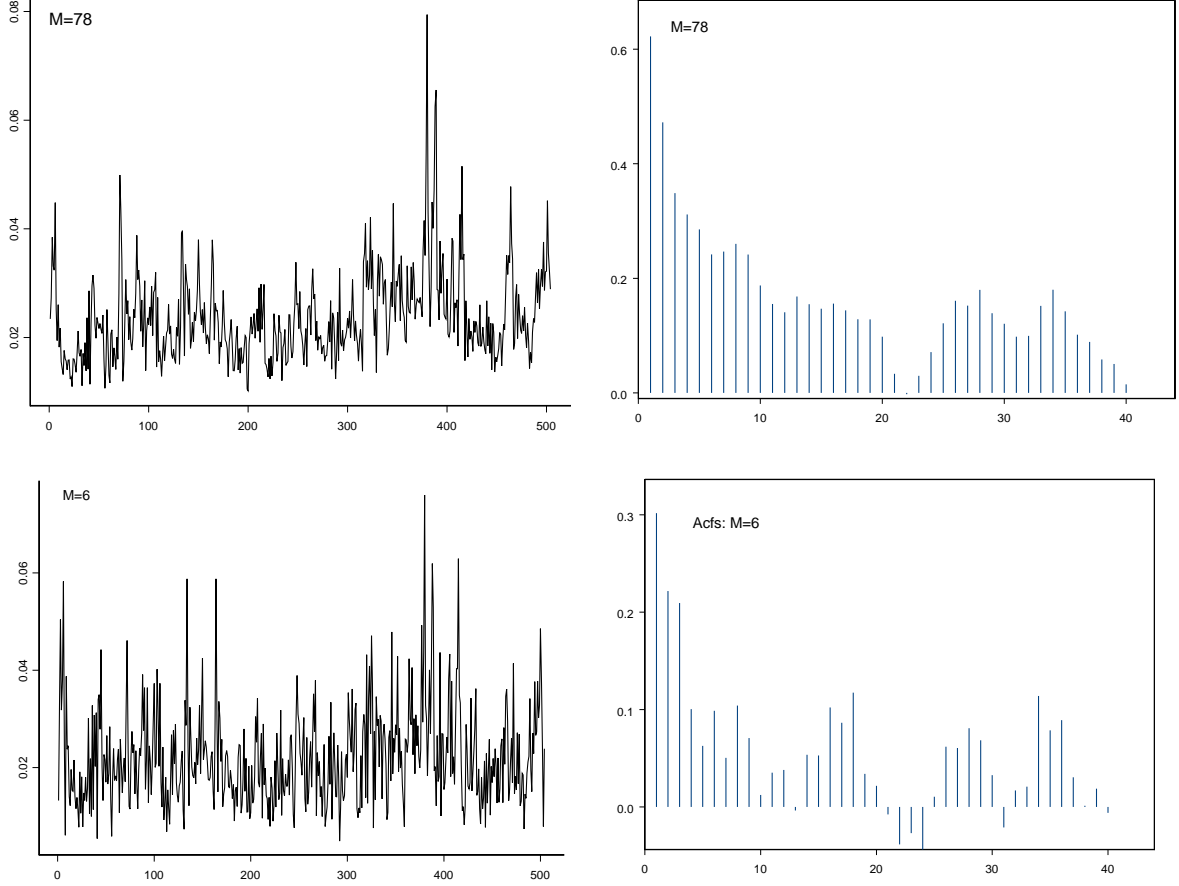


Figure 1: *Realised absolute variation computed using 5 and 60 minutes INTEL returns and their autocorrelation function.*

2.2. Realised variance

The realised quadratic variation process is defined as

$$[Y_\delta]_t^{[2]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} y_{j,\delta}^2, \quad \text{so as } \delta \downarrow 0 \quad [Y_\delta]_t^{[2]} \xrightarrow{p} [Y]_t.$$

For daily series, the realised variance is defined by the summation of the squares of these M intra-period returns and, under semimartingales, as $M \rightarrow \infty$ then

$$[Y_M]_i^{[2]} = \sum_{j=1}^M y_{j,i}^2 \xrightarrow{p} \nu_i^2 = [Y]_{ih} - [Y]_{(i-1)h},$$

so the realised variance can be used as an estimator of the actual variance.

In Barndorff-Nielsen and Shephard (2002), Barndorff-Nielsen and Shephard (2003) and Barndorff-Nielsen and Shephard (2004b) the previous theory has been extended to a Central Limit Theorem (CLT). In these papers the CLT is presented under somewhat restrictive assumptions. Recently Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) give weaker conditions on the log-price process which ensure that the CLT holds. For the SV model (1), when $\delta \downarrow 0$

$$\frac{\delta^{-1/2}([Y_\delta]_t^{[2]} - [Y]_t)}{\sqrt{2 \int_0^t \sigma_s^4 ds}} \xrightarrow{L} N(0, 1),$$

under the assumptions that A_t is of locally bounded variation, $\int_0^t \sigma_u^2 du < \infty$ and that σ_t is càdlàg.

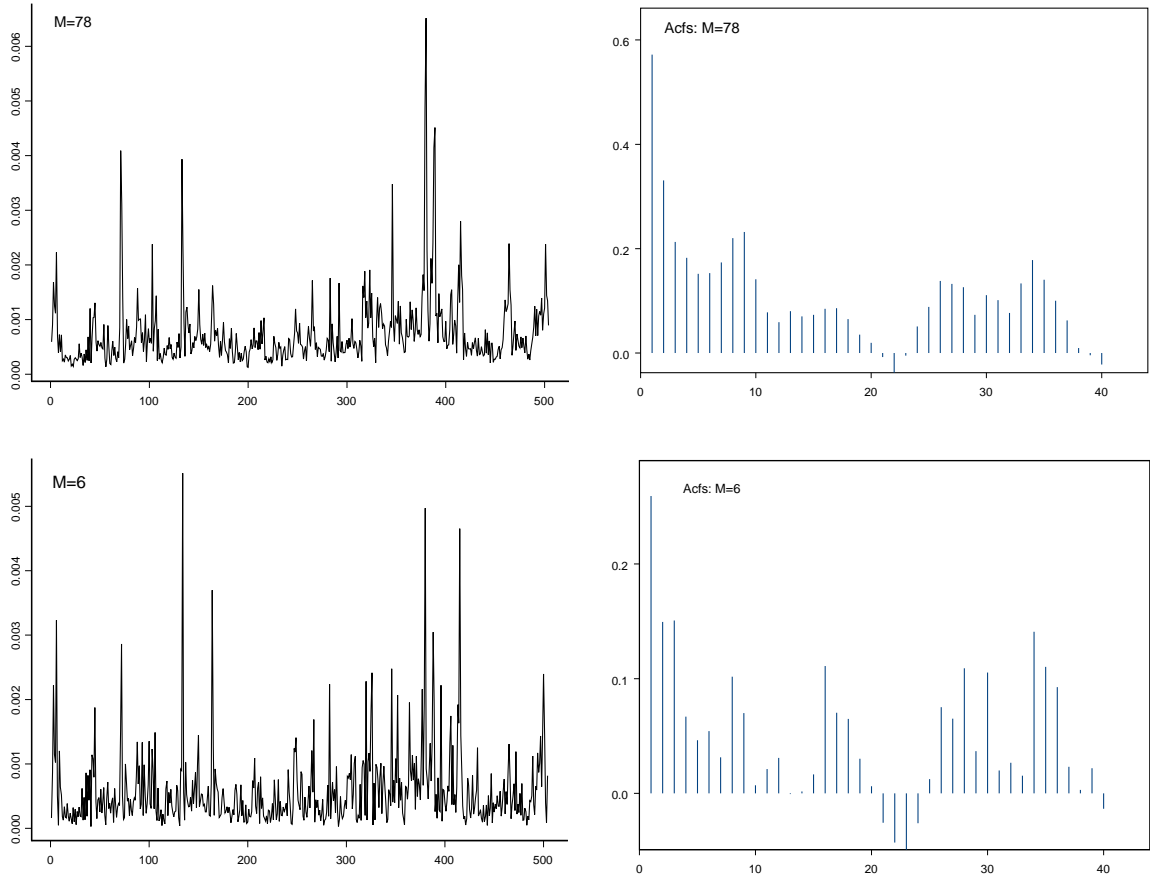


Figure 2: *Realised variance series computed using 5 and 60 minutes INTEL returns and its autocorrelation function.*

Figure 2 illustrates the time series of realised variances for the INTEL dataset. In the top part of this figure the realised variances and their ACF were computed using $M=78$, which corresponds to using five minutes returns data based on the INTEL dataset. The realised variances computed using $M=6$, sixty minutes returns, and the correspondent ACF are illustrated in the bottom part of the figure. It can be observed that the realised variance series calculated using five minutes returns ($M=78$) is much less jagged than the series calculated using 60 minutes returns ($M=6$). The correlograms have

the usual slow decay behaviour and it starts at quite a low level for $M=6$. If we compare the ACF of the realised absolute variation with the one of the realised variance, we can notice that the one of the realised absolute variation is marginally stronger.

3. Estimations with Realised Absolute Variation

Absolute returns are well known to have desirable empirical properties, nevertheless further research is needed about their use as estimators of integrated spot volatility. In this section we will set a model and estimate integrated spot volatility using realised absolute variation. Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004) did a similar analysis for realised variance but absolute returns also need to be studied.

When estimating $\nu_i^{[1]}$ with realised absolute variation, the variance of the error is quite high even when using large values of M , so a model should be established. A linear model for $\nu_i^{[1]}$ can be defined as

$$\nu_{s:p}^{[1]} = cE(\nu_{s:p}^{[1]}) + A[Y_M]_{s:p}^{[1]}$$

where

$$[Y_M]_{s:p}^{[1]} = \frac{1}{\sqrt{M}\mu_1} \left(\sum_{j=1}^M |y_{j,s}|, \sum_{j=1}^M |y_{j,s+1}|, \dots, \sum_{j=1}^M |y_{j,p}| \right)',$$

and

$$\nu_{s:p}^{[1]} = \left(\nu_s^{[1]}, \nu_{s+1}^{[1]}, \dots, \nu_p^{[1]} \right)'$$

We know that

$$\sqrt{\frac{M}{h}} \left([Y_M]_{s:p}^{[1]} - \nu_{s:p}^{[1]} \right) | \nu_{s:p}^2 \xrightarrow{L} N \left(0, \left(\frac{\pi}{2} - 1 \right) A \text{diag}(\nu_{s:p}^2) A' \right).$$

Therefore, we can obtain for the realised absolute variation that

$$\sqrt{\frac{M}{h}} \left(A[Y_M]_{s:p}^{[1]} - A\nu_{s:p}^{[1]} \right) | \nu_{s:p}^2 \xrightarrow{L} N \left(0, \left(\frac{\pi}{2} - 1 \right) A \text{diag}(\nu_{s:p}^2) A' \right).$$

Assuming that the realised absolute variations are a covariance stationary process, the weighted least square estimator gives in this case that

$$\begin{aligned} \hat{c} &= (I - \hat{A})\iota \\ \hat{A} &= \text{Cov}(\nu_{s:p}^{[1]}) \left(\text{Cov}(\nu_{s:p}^{[1]}) + \left(\frac{\pi}{2} - 1 \right) \frac{E(\nu_i^2)h}{M} I \right)^{-1}. \end{aligned}$$

We can obtain the estimators in a model free or a model based manner. We first look at the model free approach.

3.1. Sample based method

When using empirical averages, if we have a covariance stationary process of realised absolute variations and an ergodic daily process, then we know that

$$\frac{1}{T} \sum_{i=1}^T [Y_M]_i^{[1]} \xrightarrow{p} E(\nu_i^{[1]})$$

and

$$\left(\frac{1}{T} \sum_{i=1}^T \sum_{j=1}^M y_{j,i}^2 \right) \xrightarrow{p} E(\nu_i^2)$$

when M and T go to infinity. With these results we can obtain the estimates, as

$$\hat{A} = \left(Cov([Y_M]_{s:p}^{[1]}) - \left(\frac{\pi}{2} - 1 \right) \frac{E(\nu_i^2)h}{M} I \right) \left(Cov([Y_M]_{s:p}^{[1]}) \right)^{-1}.$$

The $Cov([Y_M]_{s:p}^{[1]})$ is easily calculated from the data, and as we are working in daily bases we set $h = 1$.

Our first results are obtained using a single value of realised absolute variation in the model, $s = p = i$. From Table 1 we can appreciate that, as expected, as M gets larger the weight given to the realised absolute variation becomes more important. The difference in the weights when using 5 minute returns compared to when using 60 minutes returns is very illustrative in this sense.

M	min	\hat{c}	\hat{A}
78	5	0.0802	0.9197
39	10	0.1413	0.8586
26	15	0.2050	0.7949
19	20	0.2513	0.7486
13	30	0.3652	0.6347
6	60	0.5536	0.4463

Table 1: *Estimated weights for the estimation of $\nu_i^{[1]}$.*

3.1.1. Models with lags and leads

Now in a dynamic approach, starting with the simplest model, let us estimate $\nu_{s:p}^{[1]}$ using one lag, one lead and the contemporaneous realised absolute variation.

When working with five minutes returns we obtain the following matrices

$$\hat{A} = \begin{pmatrix} 0.86617 & 0.07172 & 0.01856 \\ 0.07172 & 0.83031 & 0.07172 \\ 0.01856 & 0.07172 & 0.86617 \end{pmatrix} \quad \hat{c} = \begin{pmatrix} 0.04354 \\ 0.02624 \\ 0.04354 \end{pmatrix},$$

so from the second row of the matrices the weights are obtained, giving the following estimator

$$\widehat{\nu_i^{[1]}} = 0.026E(\nu_i^{[1]}) + 0.071[Y_{78}]_{i-1}^{[1]} + 0.830[Y_{78}]_i^{[1]} + 0.071[Y_{78}]_{i+1}^{[1]}.$$

Observe that much more weight is given to the contemporaneous realised absolute variation than to the lag and lead. Nevertheless, if we compare this estimator to the previous one with no lags or leads, less weight is given to the expectation and the weight given as a total to the realised absolute variation (lag, contemporaneous and lead) has increased.

When the realised absolute variation is calculated with sixty minutes returns, the estimator is

$$\widehat{\nu_i^{[1]}} = 0.331E(\nu_i^{[1]}) + 0.159[Y_6]_{i-1}^{[1]} + 0.349[Y_6]_i^{[1]} + 0.159[Y_6]_{i+1}^{[1]}.$$

Again we can appreciate that as a smaller value of M is used, the less weight is given to the realised absolute variation. By using one lag and one lead, the weight in the expectation of $\nu_i^{[1]}$ decreases, and the weight given in total to the realised absolute variation increases compared to the estimation without lags or leads, independently of the value of M.

The number of lags and leads can be increased in the model. When using two lags and two leads for the estimation, we obtained a 5x5 matrix for \hat{A} and a 5x1 vector for \hat{c} . Using their third rows, the following models are obtained when using five and sixty minutes returns consecutively,

$$\widehat{\nu_i^{[1]}} = 0.015E(\nu_i^{[1]}) + 0.016[Y_{78}]_{i-2}^{[1]} + 0.062[Y_{78}]_{i-1}^{[1]} + 0.826[Y_{78}]_i^{[1]} + 0.062[Y_{78}]_{i+1}^{[1]} + 0.016[Y_{78}]_{i+2}^{[1]}$$

$$\widehat{\nu_i^{[1]}} = 0.253E(\nu_i^{[1]}) + 0.073[Y_6]_{i-2}^{[1]} + 0.133[Y_6]_{i-1}^{[1]} + 0.332[Y_6]_i^{[1]} + 0.133[Y_6]_{i+1}^{[1]} + 0.073[Y_6]_{i+2}^{[1]}$$

Here, again, the larger the value of M, the bigger the weight given to the contemporaneous realised absolute variation. Also, the closer the lags or the leads are to the contemporaneous realised absolute variation, the bigger the weight given to them. Now, the weights of the unconditional expectation are even smaller because the actual volatility can be better explained with the lags.

In Figure 3 the weights of a model using three lags, three leads and the contemporaneous realised absolute variation are plotted for different values of M. It shows how quickly the weights focus on the contemporaneous observation of the realised absolute variation as M increases.

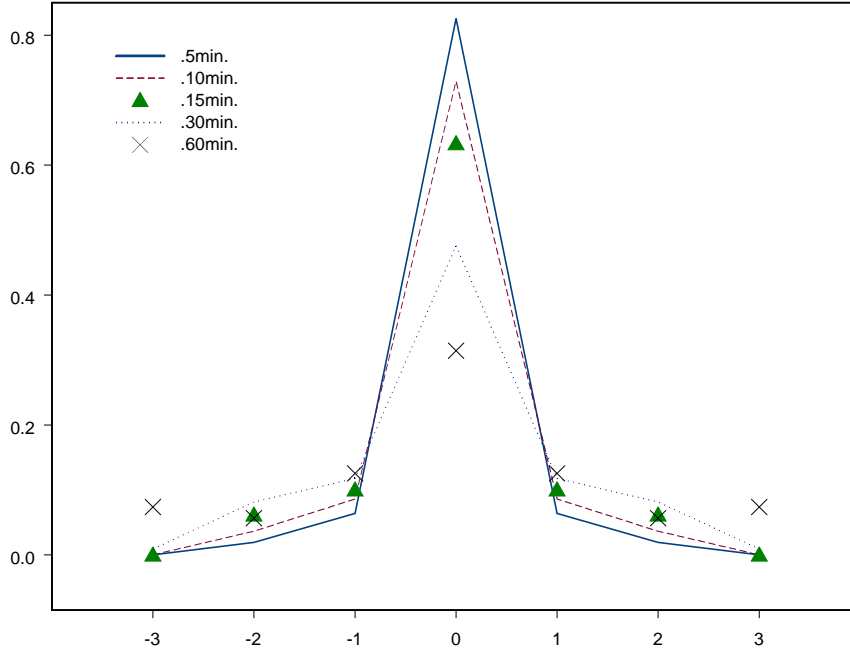


Figure 3: *Weights for estimating actual volatility using 3 lags and 3 leads for different values of M .*

3.1.2. Logarithms

A similar analysis can be established using log-realised absolute variations. The relevant asymptotic results obtained with the Delta method is the following

$$\sqrt{\frac{M}{h}} \left(A \log([Y_M]_{s:p}^{[1]}) - A \log(\nu_{s:p}^{[1]}) \right) | \nu_{s:p}^2, \nu_{s:p}^{[1]} \xrightarrow{L} N \left(0, \left(\frac{\pi}{2} - 1 \right) A E \begin{pmatrix} \frac{\nu_s^2}{(\nu_s^{[1]})^2} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \frac{\nu_p^2}{(\nu_p^{[1]})^2} \end{pmatrix} A' \right)$$

so now we can use the fact that

$$\left(\frac{1}{T} \sum_{i=1}^T \frac{\sum_{j=1}^M y_{j,i}^2}{\left(\frac{1}{\sqrt{M}\mu} \sum_{j=1}^M |y_{j,i}| \right)^2} \right) \xrightarrow{p} E \left(\frac{\nu_i^2}{(\nu_i^{[1]})^2} \right)$$

and that

$$\frac{1}{T} \sum_{i=1}^T \log[Y_M]_i^{[1]} \xrightarrow{p} E(\log \nu_i^{[1]})$$

to obtain the estimators.

This allows us to construct the estimate

$$\widehat{\log \nu_{s:p}^{[1]}} = \hat{c} E(\log \nu_{s:p}^{[1]}) + \hat{A} \log[Y_M]_{s:p}^{[1]},$$

and given that the realised absolute variations are a covariance stationary process, then the weighted least squares estimator set

$$\hat{c} = (I - \hat{A})\iota$$

and

$$\hat{A} = Cov(log\nu_{s:p}^{[1]}) \left(Cov(log\nu_{s:p}^{[1]}) + \left(\frac{\pi}{2} - 1 \right) \frac{h}{M} E \left(\frac{\nu_i^2}{(\nu_i^{[1]})^2} \right) I \right)^{-1}.$$

First, by setting $s = p = i$, i.e. using just the contemporaneous realised absolute variation, and using logarithms, the results shown in Table 2 are obtained. As before, for smaller values of M used, the weight given to the realised absolute variation gets smaller as well. Compared to Table 1, when using logarithms and M is small, higher values for the weight of the log-realised absolute variation are found. Notice the weights increase just when M is small.

M	min	\hat{c}	\hat{A}
78	5	0.0862	0.9137
39	10	0.1465	0.8534
26	15	0.2045	0.7954
19	20	0.2633	0.7366
13	30	0.3337	0.6662
6	60	0.4718	0.5281

Table 2: *Estimated weights for the estimation of $log\nu_i^{[1]}$.*

When using one lag and one lead in the logarithm model, the same conclusions can be obtained. For five minutes returns we get the model

$$\widehat{\nu_i^{[1]}} = 0.032E(log\nu_i^{[1]}) + 0.067log[Y_{78}]_{i-1}^{[1]} + 0.833log[Y_{78}]_i^{[1]} + 0.067log[Y_{78}]_{i+1}^{[1]},$$

and for sixty minutes

$$\widehat{\nu_i^{[1]}} = 0.316E(log\nu_i^{[1]}) + 0.101log[Y_6]_{i-1}^{[1]} + 0.479log[Y_6]_i^{[1]} + 0.101log[Y_6]_{i+1}^{[1]}.$$

In Figure 4 the weights of the log-model using three lags, three leads and the contemporaneous observation are shown for different values of M. When using logarithms the weights given for small values of M are much higher than those obtained for the estimator based on the raw realised absolute variation (compare to Figure 3).

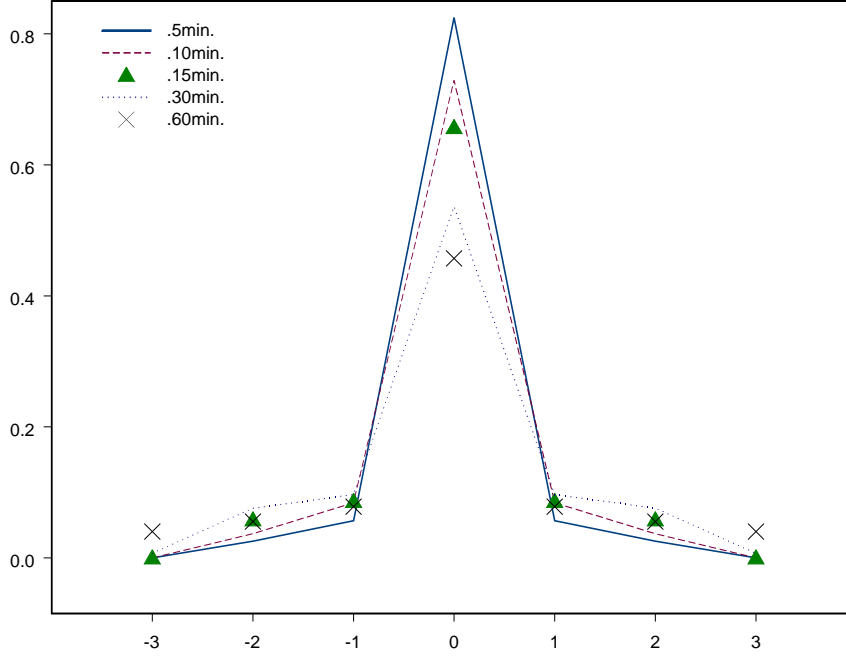


Figure 4: *Weights for estimating $\log(\nu_i^{[1]})$ using 3 lags and 3 leads for different values of M .*

3.2. Model based estimation

In order to construct a model based estimator of the actual volatility, we need to derive the first and second moments of $\nu_i^{[1]}$. If ξ is the mean of σ_t , ω is its variance, and r is its autocorrelation function, it is known that

$$E(\nu_i^{[1]}) = h\xi, \quad Var(\nu_i^{[1]}) = 2\omega^2 r_h^{**}, \quad \text{and} \quad Cov\{\nu_i^{[1]}, \nu_{i+s}^{[1]}\} = \omega^2 \diamond r_{hs}^{**}$$

where

$$\diamond r_h^{**} s = r_{s+h}^{**} - 2r_s^{**} + r_{s-h}^{**}, \quad r_t^* = \int_0^t r_u du,$$

$$\text{and} \quad r_t^{**} = \int_0^t r_u^* du.$$

So with the second order properties of σ_t , the second order properties of $\nu_i^{[1]}$ can be fully established. In a stochastic volatility model context, realised absolute variation can be written as

$$[Y_M]_i^{[1]} = \nu_i^{[1]} + u_i,$$

so

$$u_i = [Y_M]_i^{[1]} - \nu_i^{[1]} = \frac{1}{\sqrt{M}\mu} \sum_{j=1}^M |\varepsilon_{j,i}| \sqrt{\nu_{j,i}^2} - \sum_{j=1}^M \nu_{j,i}^{[1]}$$

where

$$\begin{aligned}
\varepsilon_{j,i} &\stackrel{iid}{\sim} N(0, 1) \\
\nu_{j,i}^2 &= \sigma_{(i-1)h + \frac{jh}{M}}^{2*} - \sigma_{(i-1)h + \frac{(j-1)h}{M}}^{2*}, \\
\nu_{j,i}^{[1]} &= \sigma_{(i-1)h + \frac{jh}{M}}^{[1]*} - \sigma_{(i-1)h + \frac{(j-1)h}{M}}^{[1]*}, \\
\sigma_t^{[1]*} &= \int_0^t \sigma_u du \quad \text{and} \quad \sigma_t^{2*} = \int_0^t \sigma_u^2 du.
\end{aligned}$$

As stated before $\sqrt{\nu_{j,i}^2} \neq \nu_{j,i}^{[1]}$, so

$$E(\sqrt{M}u_i | \nu_i^{[1]}) \neq 0.$$

Nevertheless, it is known that $\sqrt{M}u_i \xrightarrow{L} MN(0, (\frac{\pi}{2} - 1)h \int_{(i-1)h}^{ih} \sigma_u^2 du)$ so

$$E(\sqrt{M}u_i | \nu_i^{[1]}) = o(1).$$

The second order properties of $[Y_M]_i^{[1]}$ can now be found. First set

$$\begin{aligned}
E([Y_M]_i^{[1]}) &= E(\nu_i^{[1]}) + E(u_i) = h\xi + \frac{1}{\sqrt{M}}o(1), \\
Var([Y_M]_i^{[1]}) &= Var(\nu_i^{[1]}) + Var(u_i) + o(1), \\
Cov([Y_M]_i^{[1]}, [Y_M]_{i+s}^{[1]}) &= Cov(\nu_i^{[1]}, \nu_{i+s}^{[1]}) + o(1).
\end{aligned}$$

Just $Var(u_i)$ is left to be found to determine these second order properties. Again from the fact that

$$\sqrt{M}u_i \xrightarrow{L} MN\left(0, \left(\frac{\pi}{2} - 1\right)h \int_{(i-1)h}^{ih} \sigma_u^2 du\right)$$

we have that

$$\begin{aligned}
Var(\sqrt{M}u_i) &= \left(\frac{\pi}{2} - 1\right)hE\left(\int_{(i-1)h}^{ih} \sigma_u^2 du\right) + o(1) \\
&= \left(\frac{\pi}{2} - 1\right)h \int_{(i-1)h}^{ih} E(\sigma_u^2) du + o(1) \\
&= \left(\frac{\pi}{2} - 1\right)h^2 E(\sigma_u^2) + o(1) \\
&= \left(\frac{\pi}{2} - 1\right)h^2 (Var(\sigma_u) + E^2(\sigma_u)) + o(1)
\end{aligned}$$

so

$$Var(\sqrt{M}u_i) = \left(\frac{\pi}{2} - 1\right)h^2(\omega^2 + \xi^2) + o(1).$$

With these results the variance of the realised variation and its covariance can be established,

$$Var([Y_M]_i^{[1]}) = \left(\frac{\pi}{2} - 1\right)\frac{h^2}{M}(\omega^2 + \xi^2) + 2\omega^2 r_h^{**} + \frac{1}{M}o(1),$$

$$Cov([Y_M]_i^{[1]}, [Y_M]_{i+s}^{[1]}) = \omega^2 \diamond r_{sh}^{**} + o(1).$$

3.2.1. Model

Similar to the realised variance case, for modelling the stochastic volatility, a process with an autocorrelation function as $r_t = \exp(-\lambda|t|)$ will be used. The process, which is the solution to the stochastic differential equation

$$d\sigma_t = -\lambda\sigma_t dt + dz_{\lambda t} \quad (3)$$

where z_t is a Levy process with non-negative increments, has the previous acf.

For this process $r_t^{**} = \lambda^{-2}\{e^{-\lambda t} - 1 + \lambda t\}$ and $\diamond r_{sh}^{**} = \lambda^{-2}(1 - e^{-\lambda h})^2 e^{-\lambda(s-1)h}$, $s > 0$; implying that the asymptotic moments are

$$\begin{aligned} E(\nu_i^{[1]}) &= \xi h, \\ Var(\nu_i^{[1]}) &= \frac{2\omega^2}{\lambda^2}(e^{-\lambda h} - 1 + \lambda h) \\ Cor\{\nu_i^{[1]}, \nu_{i+s}^{[1]}\} &= \frac{(1 - e^{-\lambda h})^2(e^{-\lambda(s-1)h})}{2(e^{-\lambda h} - 1 + \lambda h)}. \end{aligned}$$

The autocorrelation model for the $\nu_i^{[1]}$ is that one for an ARMA(1,1), so the following linear state-space representation can be used:

$$\begin{aligned} [Y_M]_i^{[1]} &= \xi h + (1 \ 0) \alpha_i + \sigma_u v_{1i} \\ \alpha_{i+1} &= \begin{pmatrix} \phi & 1 \\ 0 & 0 \end{pmatrix} \alpha_i + \begin{pmatrix} \sigma_\sigma \\ \sigma_\sigma \theta \end{pmatrix} v_{2i} \end{aligned}$$

where

$$\alpha_i = \nu_i^{[1]} - \xi h \quad \text{and} \quad u_i = \sigma_u v_{1i}.$$

Here v_i is a zero mean, white noise sequence with an identity covariance matrix; ϕ , θ , and σ_σ are the autoregressive root, the moving average root and the variance of the innovation of the process. Finally σ_u^2 is found from the asymptotic result for the $Var(u_i)$. The Kalman filter can be used for predicting and smoothing $\nu_i^{[1]}$.

Again, just one of the previous volatility models is too simple to fit the long-range dependence financial time series have, so processes will be superposed to describe σ_u . In this case, we have

$$\sigma_t = \sum_{j=1}^J \sigma_t^{(j)} \quad \text{and} \quad \sum_{j=1}^J w_j = 1$$

where w_j are the weights that must be larger or equal to zero and $\sigma_t^{(j)}$ are the processes with memory λ_j .

Using the previous model $\nu_i^{[1]}$ can be estimated for the INTEL dataset. The model was fitted using five, fifteen and sixty minutes returns, determining the number of processes needed in the superposition and estimating the parameters.

When using five minutes returns, the empirical ACF is best described with the superposition of three processes (Figure 5); adding an additional process to obtain a superposition of four processes does not contribute. Yet two processes were not enough to describe the ACF, so this third component is essential. From Table 3 it can be seen how two of the processes have a lot of memory (low values of λ) and the other one has very low memory (high value of λ). There is an important difference in the Box-Pierce statistic of the models based on one process and two processes. It can be noticed that the model-based estimator with only one process do not give a good fit. It is until the superposition of three processes is used when the long-range dependence of the data is picked up.

The model is fitted as well for smaller value of M , first for fifteen minutes returns. Again the superposition of three processes is the model that gives the best fit. Two of the components have small values of λ and just one a large value explaining the long-range dependence. The same happens when using sixty minutes returns; the best fit was found when using three processes. Also, two of the components have persistent variance (low value of λ), and the fit does not improve when changing to four the number of processes, but it does when changing from two to three.

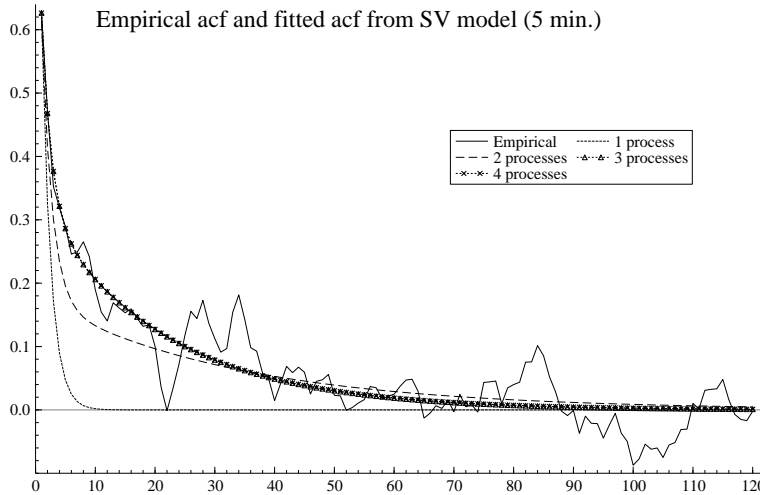


Figure 5: Using $M=78$, ACF of $[Y_M]_i^{[1]}$ and the fitted version for various values of J .

	ξ	ω^2	λ_1	λ_2	λ_3	λ_4	w_1	w_2	w_3	L_Q	BP_{20}
$J = 1$	3.41	5.39	0.64	-	-	-	1.00	-	-	-11,191	53.61
$J = 2$	3.48	5.08	0.03	0.57	-	-	0.18	-	-	-10,834	13.41
$J = 3$	3.48	7.02	0.04	0.61	291.9	-	0.15	0.26	-	-10,804	10.88
$J = 4$	3.49	7.05	0.04	0.61	57.3	291.9	0.19	0.32	0.42	-10,804	10.89

Table 3: Fit of the superposition of J volatility processes for a SV model based on realised absolute variation using $M=78$. Parameters, Quasi-Likelihood (L_Q) and Box-Pierce (BP) statistic.

3.2.2. Comparison

A comparison can be made of the smoothed series from the model free and the model based approach. Table 4 gives the correlation between the model based estimators and the model free estimators with different number of lags and leads and for different values of M . As the number of

lags and leads increases they become more closely correlated, and also as M increases the connection between the estimators becomes stronger. In Figure 6 the first 200 estimated values of $\nu_i^{[1]}$ from the time series are shown. The time series correspond to the model free estimators using two lags and two leads and the smoothed model based estimator. It can be appreciated that there exists a close connection between these two estimators.

	5min.	15min.	60min.
$s = p$	0.9587	0.8941	0.7838
1 lag 1 lead	0.9805	0.9612	0.9508
2 lags 2 leads	0.9811	0.9692	0.9772
logarithms	0.9586	0.8925	0.7702
logs 1 lag 1 lead	0.9781	0.9507	0.8836

Table 4: *Correlation between the model free and model based smoothers.*

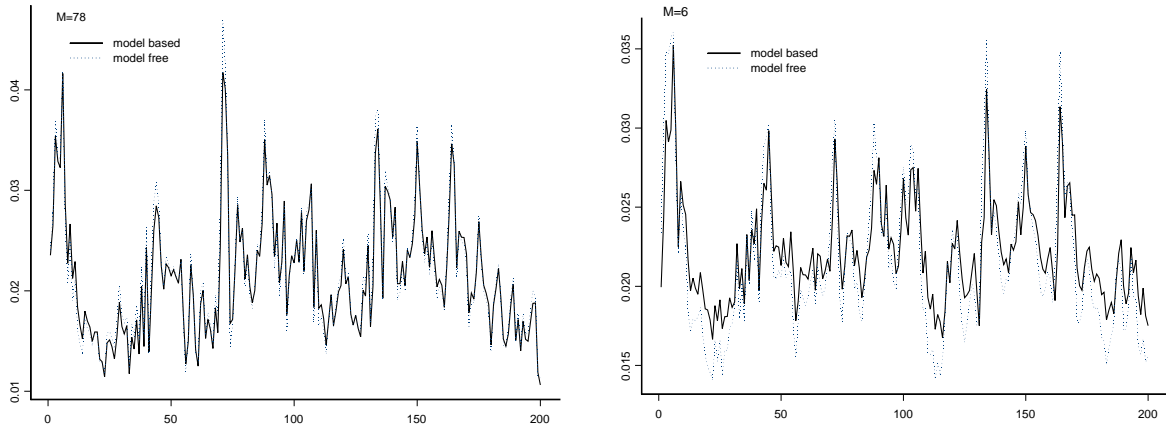


Figure 6: *Model free using 2 lags and 2 leads and model based estimators of $\nu_i^{[1]}$ for $M=78$ and $M=6$.*

3.3. Absolute returns vs squared returns

It is well known that absolute returns can give better results than squared returns for the estimation of integrated variance. If we perform the equivalent previous modelling, filtering and smoothing but for realised variance (as in Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004)) using the INTEL database, realised absolute variation seems to be more robust than realised variance. This is consistent with Forsberg and Ghysels (2006) who provide a theoretical explanation for the fact that realised absolute variation outperforms realised variance when estimating integrated spot volatility and integrated variance respectively.

Using population moments, these authors prove that the use of absolute values yields a higher persistence. When using high-frequency data for the estimation, the bias due to sampling errors is much smaller with absolute returns (realised absolute variation) than with squared returns (realised variance). The authors studied the effect of discrete jumps on the log-price process and concluded that absolute returns were immune to their presence. They highlighted three reasons that made absolute returns more persistent than squared returns: 1) desirable population predictability features, 2) better sampling error behaviour and 3) immunity to jumps.

From this we can conclude that realised absolute variation is a better estimator of integrated spot volatility than realised variance is of integrated variance. Nevertheless, the main interest is to estimate integrated variance. Let us compare the asymptotic error distributions for the realised variance process and the squared realised absolute variation process, as well as their logarithmic transformations. In the first case we compare

$$[Y_\delta]_t^{[2]} - \int_0^t \sigma_u^2 du \xrightarrow{L} MN\left(0, 2/\delta \int_0^t \sigma_u^4 du\right)$$

and

$$\left([Y_\delta]_t^{[1]}\right)^2 - \left(\int_0^t \sigma_u du\right)^2 \xrightarrow{L} MN\left(0, \frac{4}{\delta} \left(\frac{\pi}{2} - 1\right) \int_0^t \sigma_u^2 du \left(\int_0^t \sigma_u du\right)^2\right)$$

therefore the following inequality needs to hold for absolute returns to give better results than square returns,

$$\frac{2}{4(\frac{\pi}{2} - 1)} \int_0^t \sigma_u^4 du > \int_0^t \sigma_u^2 du \left(\int_0^t \sigma_u du\right)^2.$$

Whether the inequality will hold is unclear, so now we will use a logarithmic transformation where

$$\log([Y_\delta]_t^{[2]}) - \log\left(\int_0^t \sigma_u^2 du\right) \xrightarrow{L} MN\left(0, \frac{2}{\delta} \frac{\int_0^t \sigma_u^4 du}{\left(\int_0^t \sigma_u^2 du\right)^2}\right)$$

and

$$\log\left(\left([Y_\delta]_t^{[1]}\right)^2\right) - \log\left(\left(\int_0^t \sigma_u du\right)^2\right) \xrightarrow{L} MN\left(0, \frac{4(\frac{\pi}{2} - 1)}{\delta} \frac{\int_0^t \sigma_u^2 du}{\left(\int_0^t \sigma_u du\right)^2}\right)$$

to obtain that

$$\frac{2}{4(\frac{\pi}{2} - 1)} \int_0^t \sigma_u^4 du > \frac{\left(\int_0^t \sigma_u^2 du\right)^3}{\left(\int_0^t \sigma_u du\right)^2}.$$

Again it is unclear whether this is true. But even if it was, we could only say that squared realised absolute variation is a better estimator of the squared integrated spot volatility than realised variance is of the integrated variance. Nevertheless that is not exactly what we are looking for as $\left(\int_0^t \sigma_u du\right)^2$ will not always equalise our object of interest, $\int_0^t \sigma_u^2 du$. Therefore we would then need to estimate the integrated variance with $\left(\int_0^t \sigma_u du\right)^2$, adding up an extra error.

Forsberg and Ghysels (2006) also set some regression models to predict ν_{t+1}^2 with ν_t^2 or with $\nu_t^{[1]}$. They obtained a better fit after using $\nu_t^{[1]}$ as regressor. Ghysels, Santa-Clara and Valkanov (2003) obtained similar empirical results.

Although it is unclear whether realised absolute variation can give better estimations of the integrated variance than realised variance, studies point towards the use of absolute returns rather than squared returns. Forsberg and Ghysels (2006) and Andersen, Bollerslev and Diebold (2003) studied bipower variation, introduced by Barndorff-Nielsen and Shephard (2004a), as estimator and predictor of integrated variance and reported that it would improve upon realised variance.

4. Conclusions

Given a SV model for the log-prices and the concepts of quadratic and power variation, the availability of high frequency data enabled us to use a time series of realised absolute variation to estimate actual volatilities. When using the raw realised absolute variations as estimator, the errors were large especially when M was small. By using the asymptotic distributions of these errors, we improved the estimations via a model based and a model free approach. Both approaches tended to give similar results when M was large and when several lags and leads were used in the model free estimator. Although absolute values are preferable than squares, realised absolute variation does not estimate the object of interest, integrated variance, so alternative objects need to be studied as multipower variation.

5. Appendix: INTEL dataset

Intel stocks are traded on the NASDAQ exchange which predominantly focuses on high technology stocks. The dataset will be constructed from the Trade and Quotes (TAQ) Database. The TAQ Database is the collection of intra-day trades and quotes for all the securities listed in all the main United States of America equity markets. All this information is available on a collection of CD-ROMs, with each month having between 3 and 10 separate CDs. Further information can be found on the NYSE website (www.nyse.com).

We will work only with transaction prices although a similar analysis can be done based on quote data. The TAQ database includes each transaction price, but we will take the last recorded price every five minutes to have regularly spaced data (last tick method, e.g. Wasserfallen and Zimmermann(1985)). We take the prices every five minutes from 9:30 a.m. (when the market opens) until 4:00 in the afternoon (when the market closes) for every working day from the first of October of 1998 to the twenty-ninth of September of 2000.

Problems with missing data and split markets can be encountered in this dataset. It consists of 39,816 observations (79 observations each day for 504 days), and there are 199 missing values. There are just seven days with missing values, and four of them have missing values because the market closed early that day. To obtain a complete database, a linear interpolation or a Brownian bridge can be used. Figure 7a shows a day with missing values and Figure 7b illustrates both methods (observations 56-59 and 61-64 are missing). The problem with the linear interpolation is that the variance of the returns will be approximately zero for that interval, so we will use a Brownian bridge.

Equity prices used to be decreed by the New York stock exchange to have to be integer multiples of $1/8$ of a Dollar until June 24 of 1997. Afterwards and until January 29 of 2001, they were integer multiples of $1/16$ of a dollar. The commission charged by dealers was a fixed number of these ticks. Whenever the price was too high, the percentage of the commission was very small. The Stock Exchange tried to maintain the prices between ten and one hundred, so when the price was too high, the share was divided into two half-priced ones. Split markets are not a problem for our work because we can always multiply the series by two from the day the share was split on. Also we are working with intra-day prices and markets are split at the end of the day, so they do not affect our returns.

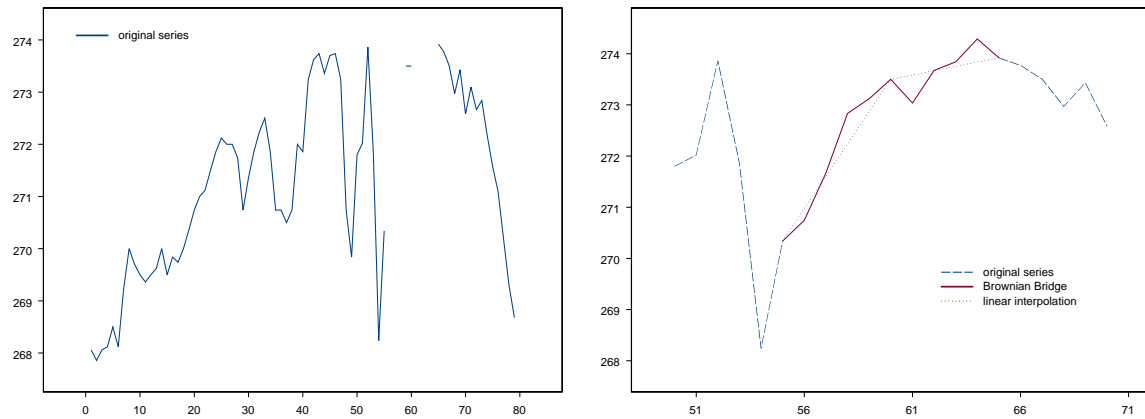


Figure 7: a) One day with missing values and b) the Brownian Bridge and the linear interpolation methods to complete the observations.

In Figure 8a the INTEL prices are plotted for the whole of our sample. We can observe how for the end of August of 2000 the prices had tripled, but afterwards, during the last month of our series, the prices fell abruptly. Figure 9a plots the intra-day prices every five minutes on a randomly selected day. With this data structure there are always 79 observations a day. In Figure 8b the five minutes returns of the dataset can be observed. The five-minutes intra-day returns, of the randomly selected day can be observed in Figure 9b.

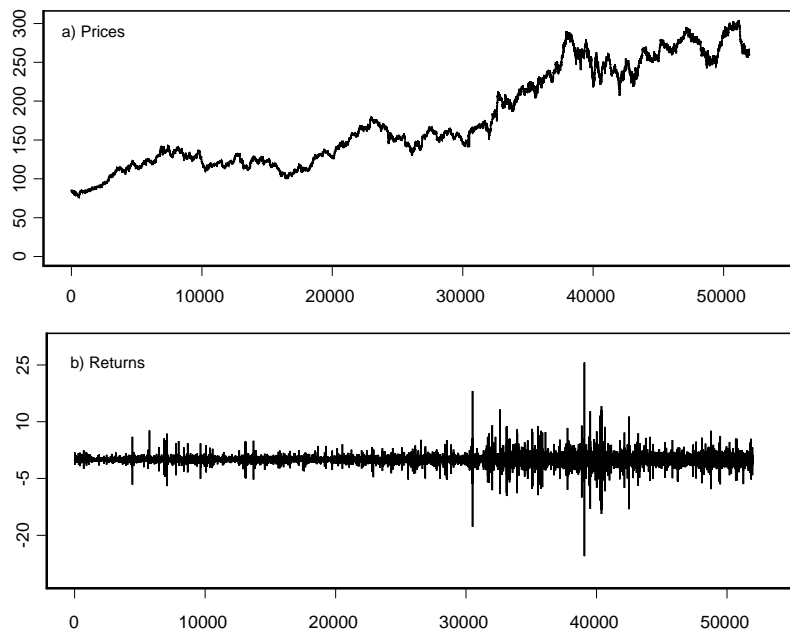


Figure 8: a) INTEL prices without split market, b) Intel five minutes returns.

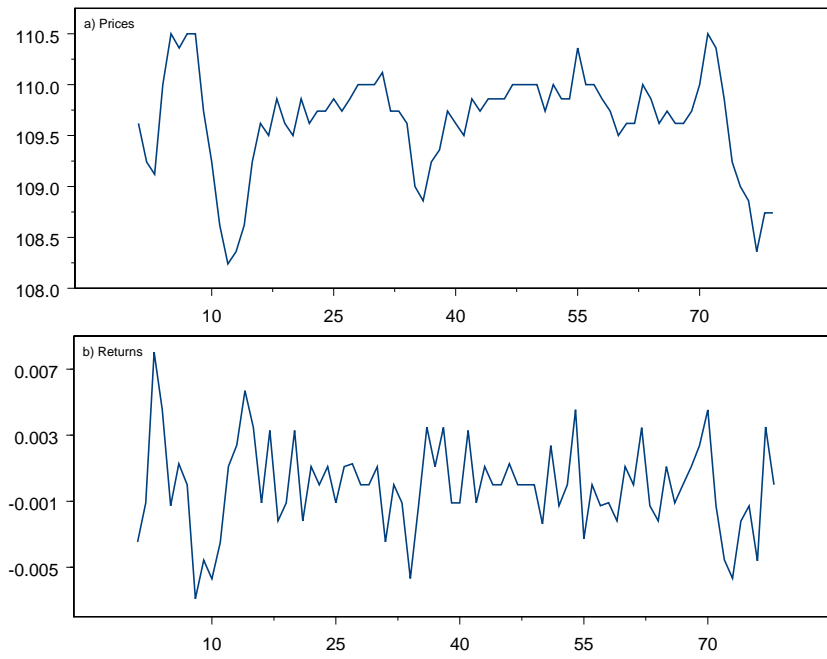


Figure 9: a) Intra-day prices every five minutes and b) the corresponding intra-day returns of a randomly selected day of the INTEL series.

References

- Andersen, T. G. and T. Bollerslev (1997). Intraday periodicity and volatility persistence in financial markets. *Journal of Empirical Finance* 4, 115-158.
- Andersen, T. G. and T. Bollerslev (1998a). Answering the skeptics: yes, standard volatility models do provide accurate forecasts. *International Economics Review* 39, 885-905.
- Andersen, T. G. and T. Bollerslev (1998b). Deutsche mark-dollar volatility: intraday activity patterns, macroeconomic announcements, and longer run dependencies. *Journal of Finance* 53, 219-265.
- Andersen, T. G., T. Bollerslev and F. X. Diebold (2003). Some like it smooth, and some like it rough: untangling continuous and jump components in measuring, modelling and forecasting asset return volatility. Unpublished paper: Economics Department, Duke University.
- Andersen, T. G., T. Bollerslev and F. X. Diebold (2005). Parametric and nonparametric measurements of volatility. In Y. Aït-Sahalia and L. P. Hansen (Eds.), *Handbook of Financial Econometrics*. Amsterdam: Elsevier Science B.V.
- Andersen, T. G., T. Bollerslev, F. X. Diebold and H. Ebens (2001). The distribution of realized stock return volatility. *Journal of Financial Economics* 61, 43-76.
- Andersen, T. G., T. Bollerslev, F. X. Diebold and P. Labys (2001). The distribution of exchange rates volatility. *Journal of American Statistical Association* 96, 42-55.
- Barndorff-Nielsen, O. E., S. E. Graversen, J. Jacod, M. Podolskij and N. Shephard (2005). A central limit theorem for realised power and bipower variations of continuous semimartingales. In Y. Kabanov and R. Lipster (Eds.), *From Stochastic Analysis to Mathematical Finance, Festschrift for Albert Shiryaev*. New York: Springer Verlag.
- Barndorff-Nielsen, O. E., S. E. Graversen, J. Jacod and N. Shephard (2006). Limit theorems for bipower variation in financial econometrics. *Econometric Theory* 22, 677-719.
- Barndorff-Nielsen, O. E., B. Nielsen, N. Shephard and C. Ysusi (2004). Measuring and forecasting financial variability using realised variance with and without a model. In A. C. Harvey, S. J. Koopman and N. Shephard (Eds.), *State Space and Unobserved Component Models: Theory and Applications. Proceedings of a Conference in Honour of James Durbin*, 205-235. Cambridge: Cambridge University Press.
- Barndorff-Nielsen, O. E. and N. Shephard (2002). Econometric analysis of realised volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society, Series B* 64, 253-280.
- Barndorff-Nielsen, O. E. and N. Shephard (2003). Realised power variation and stochastic volatility. *Bernoulli* 9, 243-265.
- Barndorff-Nielsen O. E. and N. Shephard (2004a). Power and bipower variation with stochastic volatility and jumps (with discussion). *Journal of Financial Econometrics* 2, 1-48.
- Barndorff-Nielsen, O. E. and N. Shephard (2004b). Econometric analysis of realised covariation: high frequency covariance, regression and correlation in financial economics. *Econometrica* 72, 885-925.
- Barndorff-Nielsen O. E. and N. Shephard (2007). Variation, jumps, market frictions and high frequency data in financial econometrics. In R. Blundell, P. Torsten and W. K. Newey (Eds.), *Advances*

in *Economics and Econometrics. Theory and Applications, Ninth World Congress. Econometric Society Monographs*. Cambridge: Cambridge University Press.

Christensen, B. J. and N. R. Prabhala (1998). The relation between implied and realized volatility. *Journal of Financial Economics* 37, 125-150.

Comte, F. and E. Renault (1998). Long memory in continuous-time stochastic volatility models. *Mathematical Finance* 8, 291-323.

Dacorogna, M. M., R. Gencay, U. A. Müller, R. B. Olsen and O. V. Pictet (2001). *An Introduction to High-Frequency Finance*. San Diego: Academic Press.

Ding, Z., C. Granger and R. Engle (1993). A long-memory property of stock market returns and a new model. *Journal of Empirical Finance* 1, 83-106.

Forsberg, L. and E. Ghysels (2006). Why do absolute returns predict volatility so well? Unpublished paper: Department of Economics, University of North Carolina.

Ghysels, E., P. Santa-Clara and R. Valkanov (2003). Predicting volatility: getting the most out of return data sampled at different frequencies. *Journal of Econometrics*. Forthcoming.

Hsieh, D. (1991). Chaos and nonlinear dynamics: Applications to financial markets. *Journal of Finance* 42, 281-300.

Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics* 8, 323-361.

Poterba, J. and L. Summers (1986). The persistence of volatility and stock market fluctuations. *American Economic Review* 76, 1124-1141.

Protter, P. (1990). *Stochastic Integration and Differential Equations: A New Approach*. New York: Springer.

Rosenberg, B. (1972). The behaviour of random variables with nonstationary variance and the distribution of security prices. Working paper 11, Graduate School of Business Administration, University of California, Berkely. Reprinted in Shephard (2005).

Schwert, G. W. (1989). Why does stock market volatility change over time? *Journal of Finance* 44, 1115-1153.

Shephard, N. (2005). *Stochastic Volatility: Selected Readings*. Oxford: Oxford University Press.

Shiryaev, A. N. (1999). *Essentials of Stochastic Finance. Facts, Models, Theory*. Advanced Series on Statistical Science & Applied Probability, Vol. 3. Singapore: World Scientific.

Taylor, S. J. (1986). *Modeling Financial Time Series*. New York: Wiley.

Taylor, S. J. and X. Xu (1997). The incremental volatility information in one million foreign exchange quotations. *Journal of Empirical Finance* 4, 317-340.

Wasserfallen, W. and H. Zimmermann (1985). The behaviour of intraday exchange rates. *Journal of Banking and Finance* 9, 55-72.

Zhou, B. (1996). High-frequency data and volatility in foreign exchange rates. *Journal of Business and Economic Statistics* 14, 45-52.